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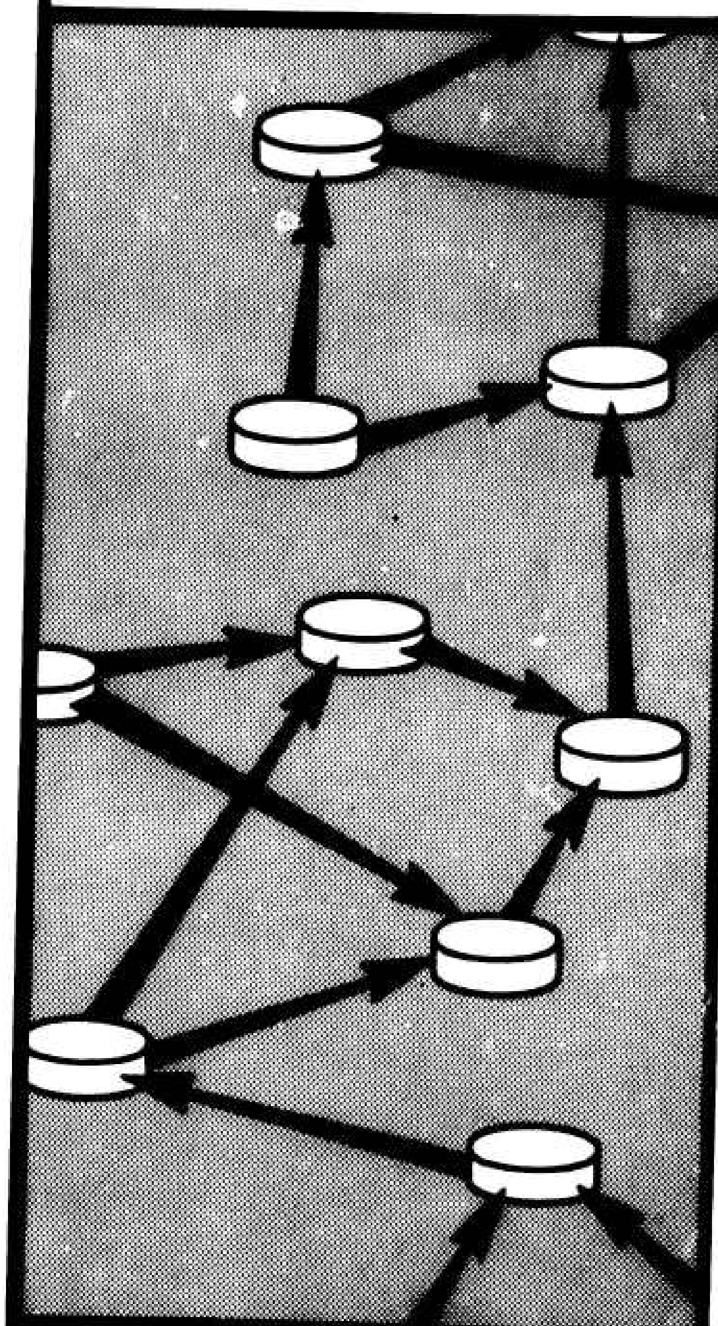
DETERMINING THE MOST VITAL LINK IN FLOW NETWORK



THE MITRE
Washington Operations

JANUARY 1971

19

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) The MITRE Corporation 1820 Dolley Madison Boulevard McLean, Virginia 22101	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Determining the Most Vital Link in a Flow Network

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

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6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 1971	18	Three (3)

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) MTP-348 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

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DD FORM 1 NOV 65 1473

Security Classification

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Networks, Maximal Flow, Interdiction						

Security Classification

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DETERMINING THE MOST VITAL LINK IN A FLOW NETWORK

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ABSTRACT

The most vital link in a single commodity flow network is that arc whose removal results in the greatest reduction in the value of the maximal flow in the network between a source node and a sink node. This paper develops an iterative labeling algorithm to determine the most vital link in the network. A necessary condition for an arc to be the most vital link is established and is employed to decrease the number of arcs that must be considered.

ACKNOWLEDGMENT

This paper was first presented at the 38th National Operations Research Society of America Meeting, Detroit, Michigan, October 28-30, 1970.

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SECTION I
INTRODUCTION

The problem of removing arcs and nodes from a network is a part of network theory that has many important and useful applications. One application would be a conflict situation where there is a logistics or communications network under attack. A defender or user of the system must know which arcs are most vital to him so that he can reinforce them against attack; while the attacker, naturally, wants to destroy those arcs whose destruction would most affect the efficiency of the system. Another application would be in helping the managers of a highway system or a transportation network to determine the effect of closing various links for repair, etc.

The problem addressed by this paper is concerned with finding the most vital link in a single commodity flow network $[N;A]$ (directed, undirected, or mixed). An arc $(x,y) \in A$ is declared to be the most vital link if its value $v(x,y)$ is at least as large as the value of every other arc in the network. The value of arc (x,y) is defined as the difference in the maximal flow values in networks $[N;A]$ and $\left[N;\left\{A-(x,y)\right\}\right]$ between some source node and some sink node. Thus, $v(x,y)$ reflects the reduction in the maximal flow attainable if arc (x,y) is removed from the network.

Wollmer has developed an algorithm for determining the most vital network link.(1) This algorithm has been employed by Durbin to determine the single most critical link in a highway system.(2)

The following sections of this paper briefly present Wollmer's method for finding the most vital link in a network and develop an improved algorithm for solving the most vital link problem. An example using the improved method is included.

SECTION II
WOLLMER'S ALGORITHM

Wollmer's algorithm for finding the most vital link in a network follows as a consequence of the following theorem. The proof of this theorem is given in reference (1).

THEOREM (Wollmer)

Suppose the network $[N;A]$ has a maximal flow value of $v(F^*)$, while the network $\left[N; \{A-(x,y)\}\right]$ has a maximal flow value of $v(F^*_{xy})$. Then, every maximal flow pattern of $[N;A]$ has at least $v(F^*) - v(F^*_{xy})$ units of flow through the arc (x,y). Moreover, there is a maximal flow pattern of $[N;A]$ which has exactly $v(F^*) - v(F^*_{xy})$ units of flow over the arc (x,y).

As Wollmer points out, "the above theorem reduces the problem to one of finding that link whose minimal flow among all maximal flow patterns is greatest." Wollmer's iterative procedure for finding this link is as follows:

<u>Step 0</u>:

Find a maximal flow pattern, F^*, in the network $[N;A]$ and let $f^*(x,y)$ be the corresponding flow in each arc $(x,y) \in A$.

Set the 'least flow' of each arc (x,y), equal to $f^*(x,y)$.

Let $f^*(p,q) = \max\limits_{(x,y) \in A} f^*(x,y)$ and go to Step 1.

<u>Step 1</u>:

Solve a maximal flow problem for the network $\left[N; \{A-(p,q)\}\right]$. Let the corresponding maximal flow pattern be denoted as F^*_{pq} and go to Step 2.

<u>Step 2</u>:

(a) Set the capacity $c(p,q)$ of arc (p,q) equal to $v(F^*) - v(F^*_{pq})$ and solve a maximal flow problem for this network (i.e., the network $[N;A]$ with $c(p,q) = v(F^*) - v(F^*_{pq})$). Call the corresponding maximal flow pattern F'. (Note F' is also a maximal flow pattern of $[N;A]$.)

(b) Next, compare the least flow of each arc (x,y) with $f'(x,y)$ and if $f'(x,y) <$ least flow of arc (x,y), replace the least flow of (x,y) with $f'(x,y)$. Reset $c(p,q)$ to its original value and go to Step 3.

Step 3:

Let $U = \max\limits_{(x,y)\,\varepsilon\, A} \left\{ \text{least flow of arc } (x,y) \right\}$.

(a) If $U \leqslant v(F^*) - v(F^*_{pq})$ terminate; (p,q) is a most vital link.

(b) If $U > v(F^*) - v(F^*_{pq})$; find an arc (p,q), such that the least flow of (p,q) equals U, and go to Step 1.

SECTION III
AN IMPROVED ALGORITHM

Wollmer's algorithm considers each arc as a candidate for the most vital link. However, a necessary condition is employed in the improved algorithm which reduces the number of arcs that must be considered explicitly as candidates.

THEOREM 1

A necessary condition for an arc (a,b) to be a most vital link is that for any maximal flow pattern in the network [N;A], the flow in arc (a,b) is at least as great as the flow over every arc in a minimal cut.

The following lemma will be useful in the proof of Theorem 1.

LEMMA 1

If $(X,\bar{X})$ is a minimal cut containing at least two arcs in a network [N;A] and if arc (x,y) is in $(X,\bar{X})$, then $(X,\bar{X}) - (x,y)$ is a minimal cut in the network $\left[N;\left|A-(x,y)\right|\right]$.

PROOF OF LEMMA 1

Suppose that $(Y,\bar{Y})$ is a minimal cut in $\left[N;\left|A-(x,y)\right|\right]$ and that $C(Y,\bar{Y}) <$ $C(X,\bar{X}) - c(x,y)$. Note that $(Y,\bar{Y})\cup(x,y)$ has to be a cut for [N;A] and that $C(Y,\bar{Y}) +$ $c(x,y) < C(X,\bar{X})$, but $(X,\bar{X})$ is a minimal cut of [N;A] and $(Y,\bar{Y})\cup(x,y)$ is a cut of [N;A]. Thus, it must be true that $C(Y,\bar{Y}) = C(X,\bar{X}) - c(x,y)$ and, hence, $(X,\bar{X}) - (x,y)$ is a minimal cut in $\left[N;\left|A-(x,y)\right|\right]$.

PROOF OF THEOREM 1

Let $(X,\bar{X})$ be a minimal cut in [N;A] and note that by hypothesis arc (a,b) is a most vital link. Assume that $f^*(a,b)<f^*(p,q) = \max_{(x,y)\in(X,\bar{X})} f^*(x,y)$ for some maximal flow pattern F^* defined in [N;A]. It will be shown that this assumption leads to a contradiction:

(a) By Lemma 1 $v(F^*_{pq}) = v(F^*) - f^*(p,q)$, by assumption $f^*(a,b) < f^*(p,q)$; therefore, $v(F^*_{pq}) = v(F^*) - f^*(p,q) < v(F^*) - f^*(a,b)$.

(b) Further, there exists a flow pattern in $\left[N;\left|A-a,b)\right|\right]$ with the value $v(F_{ab}) = v(F^*) - f^*(a,b)$, and hence, the maximal flow value, $v(F^*_{ab})$, in $\left[N;\left|A-(a,b)\right|\right]$, must satisfy $v(F^*_{ab}) \geqslant v(F^*) - f^*(a,b)$.

(c) Conditions (a) and (b) imply that $v(F^*_{ab}) > v(F^*_{pq})$; but this leads to a contradiction, since, by assumption, (a,b) is a most vital link, and by the definition of a most vital link, it follows that: $v(F^*_{ab}) \leq v(F^*_{xy})$, for all $(x,y) \in A$. Q.E.D.

Theorem 1 guarantees that those arcs whose flow is less than the largest flow through an arc in a minimal cut for some maximal flow pattern in [N;A] need not be considered as candidates for the most vital link.

SECTION IV
A LABELING SCHEME

A labeling scheme is employed in the algorithm for finding a most vital link from the set of candidate arcs. At the outset of the labeling, it is assumed that there is a maximal flow pattern, F^*, defined for the network $[N;A]$. Suppose that arc (a,b) is an arc from the candidate set (this set is made up of those arcs that satisfy the necessary condition of Theorem 1). Next, an attempt is made to label from node a to node b without using the arc (a,b). The labeling (and backtracking) rules to be used are similar to those employed by Ford and Fulkerson in their algorithm for the solution of the maximal flow problem.[3] The labeling and flow changing rules are:

Let node a be labeled with $(-, \epsilon(a) = \infty)$. At a general step, suppose the node x is labeled $(z\pm, \epsilon(x))$ and that the nodes x and y are connected by some arc. Then, node y may be labeled if either of the following situations occur:

(a) $f(y,x) > 0$; then node y is labeled $(x^-, \epsilon(y))$ where $\epsilon(y) = \min(\epsilon(x), f(y,x))$; or

(b) $f(x,y) < c(x,y)$; then node y is labeled $(x^+, \epsilon(y))$ where $\epsilon(y) = \min(\epsilon(x), c(x,y) - f(x,y))$.

The labeling process is continued until either node b is labeled (breakthrough), or node b is unlabeled, in which case no more labeling is possible (non-breakthrough). If breakthrough occurs, node b must have a label of the form $(q^\pm, \epsilon(b))$ for some node q. Likewise, node q has a label $(r^\pm, \epsilon(q))$ and node r has a label $(p^\pm, \epsilon(r))$, etc. Thus a series of nodes (and a path of arcs) starting with node b and ending with node a is defined.

Let $\epsilon = \min[\epsilon(b), f(a,b)]$. For each arc (x,y) on this path, change the flow as follows:

(a) If node y has a label $(x^+, \epsilon(y))$, replace $f(x,y)$ by $f(x,y) + \epsilon$.

(b) If node y has a label $(x^-, \epsilon(y))$, replace $f(y,x)$ by $f(y,x) - \epsilon$.

Finally, decrease the flow in arc (a,b) by ϵ units.

As Ford and Fulkerson have shown, the labeling scheme must end in one of two mentioned states: breakthrough is achieved or breakthrough is not achieved. The following results indicate what can be deduced when either of these states occur at the end of the labeling process.

THEOREM 2

Let F^* be a maximal flow pattern in the network $[N;A]$. Let (a,b) be an arc in $[N;A]$ and use the labeling scheme to label from node a to node b without using arc (a,b). If breakthrough occurs, the value of arc (a,b) is less than or equal to $f^*(a,b) - \epsilon$. If breakthrough does not occur, the value of arc (a,b) is equal to $f^*(a,b)$.

PROOF

If breakthrough occurs, an alternate maximal flow pattern is found by making an ϵ change of flow in the path established by the labeling and decreasing the flow in (a,b) by ϵ. Then $v(F^*_{ab}) \geq v(F^*) - f^*(a,b) + \epsilon$, and, since $v(a,b) = v(F^*) - v(F^*_{ab}) \leq v(F^*) - \left[v(F^*) - f^*(a,b) + \epsilon\right]$, and $\epsilon > 0$, it follows that $v(a,b) \leq f^*(a,b) - \epsilon < f^*(a,b)$.

Assume that breakthrough does not occur, then by the nature of the labeling rules, there is no maximal flow pattern in $[N;A]$ with less than $f^*(a,b)$ units of flow over arc (a,b), and, hence, by Wollmer's Theorem, $v(a,b) = f^*(a,b)$.

Q.E.D.

Utilizing these results, the following algorithm can be employed to locate a most vital link in a network:

Step 0:

(a) Find a maximal flow pattern, F^*, in the network $[N;A]$ and let $(X,\bar{X})$ be a minimal cut. Let $U^* = \max_{(x,y)\epsilon(X,\bar{X})} c(x,y)$ or, alternately, $U^* = \max_{(x,y)\epsilon(X,\bar{X})} f^*(x,y)$.

(b) Note those arcs $(x,y) \epsilon A$ for which $f^*(x,y) \geq U^*$ and store these arcs in a list; these arcs form the candidate set, S. For each arc in this set, define an upper bound as $U(x,y) = f^*(x,y)$.

Step 1:

Let $U(a,b) = \max_{(x,y)\epsilon S} U(x,y)$ and set $f(x,y) = f^*(x,y)$ for all arcs in $[N;A]$.

Step 2:

Use the labeling rules to label from node a to node b without using arc (a,b).

Step 3.

(a) If breakthrough occurs, use the backtracking and flow changing rules, replace $f(x,y)$ by the resultant flow in each $(x,y) \in A$, and if $f(a,b) > 0$, repeat Step 2.[†]

(b) Otherwise, replace $U(a,b)$ with $f(a,b)$ and if $U(a,b) \geq \max_{(x,y) \in S} U(x,y)$ terminate; arc (a,b) is a most vital link.

Otherwise, replace $f^*(x,y)$ with $f(x,y)$ for all $(x,y) \in A$ and go to Step 1.

By the use of this algorithm, a maximal flow pattern is always maintained and if there is no breakthrough for the arc (a,b) and $U(a,b) \geq U(x,y)$ for all arcs (x,y) of the candidate set S, the arc (a,b) can be declared a most vital link, i.e., non-breakthrough implies that $v(a,b) = f^*(a,b) = U(a,b)$ and, thus, $v(a,b) \geq U(x,y) \geq v(x,y)$ for $(x,y) \in S$, and by Theorem 1 the most vital link of $[N;A]$ is an element of S.

The other aspects of the method that must be considered are (1) is the method finite, and (2) is there a way to locate alternative optimal solutions? The finiteness of the procedure follows, since there can be only a finite number of arcs in the candidate set, and there is a finite number of labelings that can be made for each arc of the set. Alternative optimal solutions are readily identified as follows: after an optimal solution has been found and other candidate arcs exist, reapply the algorithm to any arc in the candidate set whose upper bound is equal to the value of the most vital link (a,b), i.e., set $U^* = U(a,b)$, delete arc (a,b) from the candidate set, and reapply the algorithm. If, upon reapplication of the algorithm, the most vital link has a value equal to U^*, then an alternative solution has been found. In the latter case, the algorithm is reapplied using a further reduced candidate set, and the procedure is continued until all alternative optimal solutions have been found.

[†] An alternative to Step 3a is to test if $f(x,y) < U(x,y)$, $(x,y) \in S$, and if $f(x,y) \geq U^*$, to replace $U(x,y)$ with $f(x,y)$, if $f(x,y) < U^*$ then arc (x,y) may be dropped from the set S. The computations are continued by either repeating Step 2 if arc (a,b) has not been dropped or returning to Step 1 if arc (a,b) has been dropped

SECTION V
EXAMPLE

Consider the flow network [N;A] shown in Figure 1.

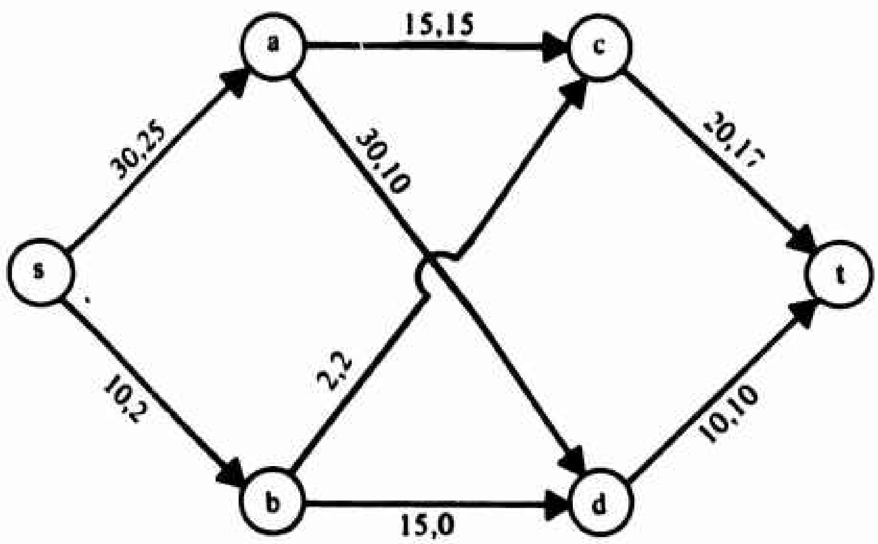

Figure 1. Maximal Flow in Sample Network with Source Node s and Sink Node t

Associated with each arc in the network is an ordered pair of numbers. The first number corresponds to the capacity of the arc and the second number corresponds to the amount of flow over the arc in some maximal flow pattern.

The improved algorithm will be employed to find all most vital links in the network [N;A].

Step 0:

(a) Figure 1 presents a maximal flow pattern F* in the network with 27 units of flow from node s to node t. The individual arc flows are:

$f^*(s,a) = 25$	$f^*(a,c) = 15$
$f^*(s,b) = 2$	$f^*(a,d) = 10$
$f^*(c,t) = 17$	$f^*(b,c) = 2$
$f^*(d,t) = 10$	$f^*(b,d) = 0$.

The corresponding minimal cut (which is unique) contains the arcs (a,c), (b,c), and (d,t). Hence,

$$U^* = \max_{(x,y) \in (X,\bar{X})} f^*(x,y) = f^*(a,c) = 15 .$$

(b) The set of candidate arcs is:

$$S = \{(s,a), (c,t), (a,c)\} .$$

<u>Step 1</u>:

Start with arc (s,a), since $U(s,a) = f^*(s,a) = \max_{(x,y) \in S} U(x,y)$.

<u>Step 2</u>:

Labeling from node s to node a without using arc (s,a) is possible over the path containing the arcs (s,b), (b,d), and (a,d). The value of $\epsilon = 8$.

<u>Step 3</u>:

(a) Since breakthrough occurred, the ϵ units of flow are removed from arc (s,a) and added to each arc in the path found in Step 2. The revised maximal flow pattern is shown in Figure 2.

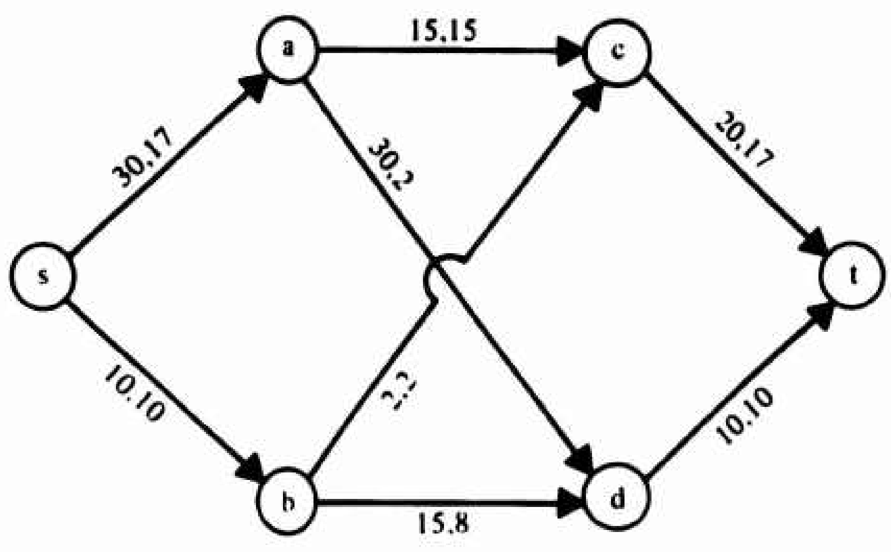

Figure 2. Revised Maximal Flow in Sample Network

Step 2:

Starting with the maximal flow pattern of Figure 2, it is not possible to label from node s to node a and non-breakthrough has occurred.

Step 3:

(b) Replace U(s,a) with the flow f(s,a) = 17. Since $U(s,a) \geq \max_{(x,y) \in S} U(x,y) = U(c,t)$, arc (s,a) is a most vital link.

Since U(c,t) = U(s,a) = 17, arc (c,t) may also be a most vital link. In order to test this hypothesis, arc (s,a) is dropped from the set S and the algorithm is reapplied starting with the maximal flow pattern in Figure 2.

Employment of the labeling rules from node c to node t results in non-breakthrough, and arc (c,t) is an alternative most vital link.

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